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# Population monotonic path schemes for simple games

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**Abstract** A *path scheme* for a game is composed of a *path*, i.e., a sequence of coalitions that is formed during the coalition formation process and a *scheme*, i.e., a payoff vector for each coalition in the path. A path scheme is called *population monotonic* if a player's payoff does not decrease as the path coalition grows. In this study, we focus on Shapley path schemes of simple games in which for every path coalition the Shapley value of the associated subgame provides the allocation at hand. Obviously, each Shapley path scheme of a game is population monotonic if and only if the Shapley allocation scheme of the game is population monotonic in the sense of Sprumont (Games Econ Behav 2:378–394, 1990). We prove that a simple game allows for population monotonic Shapley path schemes if and only if the game is balanced. Moreover, the Shapley path scheme of a specific path is population monotonic if and only if the first winning coalition that is formed along the path contains every minimal winning coalition. We also show that each Shapley path scheme of a simple game is population monotonic if and only if the set of veto players of the game is a winning coalition. Extensions of these results to other efficient probabilistic values are discussed.

**Keywords** Cooperative games · Simple games · Population monotonic path schemes · Population monotonic allocation schemes · Coalition formation · Probabilistic values

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## 1 Introduction

In many real life contexts, ranging from the formation of pre/post-electoral coalitions of parties to the formation of mergers and partnerships between firms, coalitions form through a sequence of binding bilateral agreements. From among the numerous examples of such coalition formation processes, we may single out the recent mergers between the banks and between the consultancy firms that are observed in many countries and the Oslo agreements between Israel and its neighbors. An important characteristic of such coalition formation processes is the effect of the sequence of agreements on the future potential agreements. For a coalition formed through bilateral agreements may grow larger because the synergy/commitment obtained by a coalition may create new agreement opportunities which are profitable both for the members of the coalition and the agent which will join the coalition. Hence, the determination of the sequences of binding bilateral agreements which will result in the exploitation of the greatest possible amount of synergy is of both theoretical and practical importance.

The coalition formation processes which end up with the formation of the grand coalition deserve particular interest. Because, first of all, in many situations (e.g., situations of increasing returns to size), the grand coalition is the unique efficient coalition structure. Second, the formation of the grand coalition among agents which have common properties (e.g., the formation of the grand coalition among leftist parties) has been the focal point of many branches of social sciences.

In this study, we will focus on the formation of the grand coalition through binding bilateral agreements in voting/government formation situations. We aim to address two important questions in this context.

(i) Which voting situations allow for the formation of the grand coalition through binding bilateral agreements?
(ii) In these situations, which agreement sequences must be followed to form the grand coalition?

We will address these questions by modeling voting situations by simple transferable utility cooperative games. In voting situations, the voters' incentive to form coalitions arises from their will to increase their power to affect the outcome of the voting process. Modeling of these situations as simple transferable utility games allows us to predict the voters' power to affect the result of voting by using appropriate values for transferable utility games. Many values have been offered for simple games as appropriate measures of voting power and the two most widely used ones are the Shapley–Shubik (1954) and Banzhaf (1965) power indices. If we assume that each voter's voting power is predicted by such an appropriate index, then the sequences of binding bilateral agreements which result in the formation of the grand coalition boils down to the notion of *population monotonic path schemes*.[1] Postponing a precise

[1] The notion of population monotonic (Shapley) path schemes is actually introduced by Cruijssen et al. (2005). This study applied the notion of population monotonic path schemes to a case in logistic and transportation in which several firms with a strong synergy potential can create savings through cooperation.

definition to the next section, a *path scheme* for a simple game is composed of a *path*, i.e., a sequence of coalitions that is formed through a sequence of binding bilateral agreements which result in the formation of the grand coalition and a *scheme*, i.e., a power index vector for each coalition in the path based on the associated subgame. A path scheme is called *population monotonic* if each player's index does not decrease as the path coalition grows. In this study, we focus on the Shapley–Shubik power index as an appropriate measure of voting power. Hence, the two questions that we address can be rephrased as

(i) Which simple games allow for population monotonic Shapley path schemes?
(ii) In these simple games, which Shapley path schemes are population monotonic?

It turns out that existence of veto players, i.e., a subgroup of voters whose unanimous agreement is necessary to pass a decision, is required for the existence of population monotonic Shapley path schemes and vice versa. Moreover, a Shapley path scheme is population monotonic if and only if the first winning coalition that is formed along the path contains every minimal winning coalition of the game. We also show that each Shapley path scheme of a game is population monotonic if and only if the set of veto players of the game is a winning coalition. We further show how to extend these results to the class of efficient probabilistic values, generalizations of the Shapley value introduced by Weber (1988).

Our study in particular provides an alternative prediction of what kind of coalitions form in voting situations which differs from the mainstream prediction of Riker (1962). Riker (1962) predicts that only minimal winning coalitions will form in equilibrium. This idea has been the conclusion of many studies in the general coalition formation literature based on the seminal noncooperative bargaining approach of Baron and Ferejohn (1989) and also the studies which analyze coalition formation in voting situations that are modeled by simple transferable utility games (TU-games) like Shenoy (1979). However, the empirical data on government/coalition formation shows that among all coalitions formed after the Second World War in European democracies only a third of them is minimal winning (Laver and Schofield 1990). This study shows that a wide spectrum of coalitions including the minimal winning ones can form as a result of binding bilateral agreements providing an alternative point of view for the analysis and the explanation of the data.

Population monotonic path schemes (PMPS) are in the same spirit as population monotonic allocation schemes (PMAS) for cooperative games, introduced by Sprumont (1990) and further analyzed in e.g., Norde and Reijnierse (2002) and Slikker et al. (2003). An allocation scheme for a cooperative game specifies how to distribute the worth of every coalition among its members and it is called population monotonic if the share of any player does not decrease as the coalition he/she belongs to grows larger. Clearly, also a PMAS's main concern is to ensure that no player is worse off with additional cooperation between players. However, a PMAS compares the allocations assigned to a coalition of players with every sub-coalition's allocation while a PMPS restricts the comparison to the allocations of path coalitions that are formed previously. In fact, the existence of a PMPS is a weaker condition for a TU-game than the existence of a PMAS because every path scheme induced by a PMAS is population monotonic. Another difference between the two notions is that each allocation provided by a PMAS has to belong to the core of the associated subgame. However, this may not be the case for a PMPS as we exemplify in our study.

The outline of this article is as follows. In Section 2, we will begin by introducing the preliminaries about TU-games with particular attention to simple games. Section 2 also formally introduces population monotonic path schemes. Section 3 presents the main results regarding the characterization of population monotonic Shapley path schemes of simple games. Section 4 discusses extensions of the results to other efficient probabilistic values.

## 2 Preliminaries

Given a nonempty, finite set of players $N$, a TU-game with player set $N$ is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A coalition is a set of players $S \subset N$ and $N$ is called the grand coalition. For any coalition $S \subset N$, $v(S)$ is called the worth of coalition $S$. We denote the set of TU-games with player set $N$ by $\mathcal{G}^N$. A TU-game $v \in \mathcal{G}^N$ is monotonic if $v(S) \geq v(T)$ for every $S, T \in 2^N$ with $T \subset S$. A player $i \in N$ is a null player in $v$ if $v(S \cup \{i\}) = v(S)$ for every $S \subset N \backslash \{i\}$. Given $v \in \mathcal{G}^N$ and $S \in 2^N$, the restriction of $v$ to $S$ (a subgame of $v$) is denoted by $v_{|S}$ and is defined by $v_{|S}(T) = v(T)$ for every $T \subset S$. A game $(N, v)$ is *convex* if a player's marginal contribution does not decrease if he joins a larger coalition, i.e., $v(T \cup \{i\}) - v(T) \leq v(S \cup \{i\}) - v(S)$ for every $i \in N$ and $S, T \subset N \backslash \{i\}$ with $T \subset S$. We denote by $\Pi(N)$ the set of permutations on the player set $N$.

The core of a TU-game $v \in \mathcal{G}^N$ is denoted by $C(v)$ and is defined as the set of efficient payoff vectors for which no coalition has an incentive to split off from the grand coalition, i.e., $C(v) = \{x \in \mathbb{R}^N \,|\, \sum_{i \in N} x_i = v(N)$ and $\sum_{i \in S} x_i \geq v(S)$ for all $S \in 2^N\}$. A TU-game which has a nonempty core is called a balanced game. In particular, convex games are balanced.

A function $F : \mathcal{G}^N \to \mathbb{R}^N$ is called a value. A value $F$ is efficient if for all $v \in \mathcal{G}^N$, $\sum_{i \in N} F_i(v) = v(N)$. $F$ is said to satisfy the null player property if for any $v \in \mathcal{G}^N$ and any null player $i \in N$ in $v$, $F_i(v) = 0$. $F$ is said to satisfy the null player out property (cf. Derks and Haller 1999) if elimination of a null player does not affect the value of the other players, i.e., $F_i(v) = F_i(v_{|N \backslash \{j\}})$ for all $i, j \in N$ and all $v \in \mathcal{G}^N$ such that $j$ is a null player in $v$ and $i \neq j$.

A TU-game $v \in \mathcal{G}^N$ is called simple if $v$ is monotonic, $v(S) \in \{0, 1\}$ for every $S \in 2^N$ and $v(N) = 1$. We denote the set of simple TU-games with player set $N$ by $\mathcal{S}^N$. Given $v \in \mathcal{S}^N$, a coalition $S \in 2^N$ is called a winning coalition if $v(S) = 1$ and is called a losing coalition if $v(S) = 0$. A winning coalition $S$ is called minimal winning if there does not exist a coalition $T \subsetneq S$ which is winning. Every simple game $v$ is characterized by its set of minimal winning coalitions, $MWC(v)$. A player $i \in N$ is a veto player in $v \in \mathcal{S}^N$ if $S \subset N$, $v(S) = 1$ implies that $i \in S$. The set of veto players of $v$ is denoted by $veto(v)$. It is readily verified that a simple game $v$ is balanced if and only if $veto(v) \neq \emptyset$.

Voting or decision-making situations in committees like parliaments can easily be modeled into the framework of simple games by representing the coalitions which possesses the necessary power to pass a decision as the winning coalitions of the game. This model enables the employment of values for simple games to measure the parties' power to affect the outcome of the voting situations at hand. Many values have been offered for simple games and studied in the literature as appropriate measures of decisional power, i.e., as *power indices*. We will shortly review the Shapley and Shubik (1954) power index that arises from the Shapley value.

The Shapley value (Shapley 1953) is one of the most important solution concepts in cooperative game theory and has been studied extensively. Shapley and Shubik (1954) proposed to use the Shapley value as a power index for voting situations in committees. For a simple game $v \in \mathcal{S}^N$ the Shapley–Shubik index $\Phi$ assigns to player $i \in N$

$$\Phi_i(v) = \sum_{\{S \subset N \setminus \{i\} | v(S)=0, v(S \cup \{i\})=1\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}. \tag{1}$$

The value assigned to each voter can be interpreted by using the sequential probabilistic interpretation of the Shapley value which stems from a procedure to form the grand coalition (which is described also by Shapley 1953) that yields the Shapley value of the game as an expected payoff of each player. In this procedure, the grand coalition $N$ is formed by introducing the players one by one and each player is assigned the marginal contribution to the worth of the coalition formed when she/he joins the set of her/his predecessors. Hence, the value assigned by Shapley–Shubik index is the probability of turning the coalition of predecessors from losing to winning when the order of arrival of players is random and all orders are equally likely. For further discussion of the importance of the Shapley value as an estimator of political power and several examples of its applications, the reader is referred to Straffin (1994) and Winter (2002). Lastly, we know (e.g., by Derks and Haller 1999) that the Shapley value satisfies the null player out property.

An allocation scheme specifies how to distribute the worth of every coalition among its members. That is an allocation scheme for the game $v \in \mathcal{G}^N$ is a vector $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ such that

$$\sum_{i \in S} x_i^S = v(S)$$

for every $S \in 2^N \setminus \{\emptyset\}$. Naturally, every efficient value for TU-games defines an allocation scheme where the allocation for every coalition is obtained by applying the value to the corresponding subgame. The allocation scheme in which the Shapley value is used as an allocation vector is called the Shapley allocation scheme.

Sprumont (1990) introduced the notion of population monotonic allocation schemes. The notion of population monotonicity requires that the share allocated to every player increases as the coalition to which he belongs grows larger. Formally, an allocation scheme $(x^S)_{S \in 2^N \setminus \{\emptyset\}}$ for the game $v \in \mathcal{G}^N$ is population monotonic if

$$x_i^S \geq x_i^T$$

for every $S, T \subset N$ such that $T \subset S$ and $i \in T$.

Observe that if $(x^S)_{S\in 2^N\setminus\{\emptyset\}}$ is a population monotonic allocation scheme (PMAS), then $x^S$ is a core element of the corresponding subgame $v_{|S}$ (cf. Sprumont 1990) for every $S \in 2^N\setminus\{\emptyset\}$.

We are now ready to introduce the notion of path schemes for TU-games.

Let $v \in \mathcal{G}^N$. A *path* consists of a sequence $\mathbb{S} = \{S_1, S_2, \ldots, S_{|N|}\}$ of coalitions such that $|S_k| = k$ for all $k \in \{1, \ldots, |N|\}$ and $S_m \subset S_{m+1}$ for all $m \in \{1, \ldots, |N| - 1\}$. A *path scheme* specifies how to distribute the worth of every coalition on the path among its members. Formally, a *path scheme* $(\mathbb{S}, (x^S)_{S\in\mathbb{S}})$ for $v$ consists of a path $\mathbb{S}$ and a vector $(x^S)_{S\in\mathbb{S}}$ such that

$$\sum_{i\in S} x_i^S = v(S)$$

for every coalition $S \in \mathbb{S}$.

A path scheme $(\mathbb{S}, (x^S)_{S\in\mathbb{S}})$ for $v \in \mathcal{G}^N$ is called *population monotonic* if it satisfies the following conditions:

- $x_i^S \geq v(\{i\})$ for all $S \in \mathbb{S}$ and $i \in S$ (individual rationality).
- $x_i^S \geq x_i^T$ for every $S, T \in \mathbb{S}$ such that $T \subset S$ and $i \in T$ (monotonicity).

A path scheme in which the Shapley value is used as an allocation vector is called a *Shapley path scheme*. Clearly, the Shapley allocation scheme of a TU-game is population monotonic if and only if all Shapley path schemes of the game are population monotonic. We will illustrate the notion of Shapley path schemes and their properties in the following example.

**Example 2.1** Let $N = \{1, 2, 3\}$ and $v \in \mathcal{S}^N$ be such that $MWC(v) = \{\{1, 2\}, \{2, 3\}\}$. The Shapley allocation scheme of $v$ is provided in Table 1.

It can easily be observed that the Shapley allocation scheme of $v$ is not population monotonic but that there are exactly two population monotonic Shapley path schemes on the paths $\{\{1\}, \{1, 3\}, N\}$ and $\{\{3\}, \{1, 3\}, N\}$, respectively.

Observe also that the game $v$ has a unique core allocation, (0, 1, 0) different from the Shapley value of $v$. So, in particular, the allocation prescribed by a (Shapley) PMPS may not belong to the core of the associated subgame.

**Table 1** The Shapley allocation scheme of $v$ in Example 2.1

| Coalition | Player 1 | Player 2 | Player 3 |
|---|---|---|---|
| {1} | 0 | – | – |
| {2} | – | 0 | – |
| {3} | – | – | 0 |
| {1,2} | $\frac{1}{2}$ | $\frac{1}{2}$ | – |
| {1,3} | 0 | - | 0 |
| {2,3} | – | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $N$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ |

## 3 Population monotonic shapley path schemes

We will begin with presenting a preliminary result which is useful in understanding the structure of population monotonic Shapley path schemes of simple games.

**Lemma 3.1** *Given a simple game* $v \in \mathcal{S}^N$*, let* $\mathbb{S} = \{S_1, S_2, \ldots, S_{|N|}\}$ *be a path of coalitions such that* $S_m = \{i_1, \ldots, i_m\}$ *for every* $m \in \{1, \ldots, |N|\}$*. Assume that the first winning coalition along the path* $\mathbb{S}$ *is* $S_k$*, i.e.,* $v(S_1) = \cdots = v(S_{k-1}) = 0$ *and* $v(S_k) = 1$*. If the Shapley path scheme* $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ *is population monotonic, then the following must hold*:

$$(R1)\Phi_{i_m}(v_{|S_p}) = 0, \text{ for all } m \in \{k+1, \ldots, |N|\} \text{ and for all } p \in \{m, \ldots, |N|\}.$$

$$(R2)\Phi_i(v_{|S_k}) = \Phi_i(v_{|S_p}), \text{ for all } p \in \{k+1, \ldots, |N|\} \text{ and for all } i \in S_k.$$

$$(R3) MWC(v_{|S_k}) = MWC(v).$$

*Proof* (R1) and (R2) On the one hand $\sum_{i\in S_p} \Phi_{i_m}(v_{|S_p}) = 1$ for all $p \in \{k, \ldots, |N|\}$ by the efficiency of the Shapley value. On the other hand, by the population monotonicity of $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$, $\Phi_i(v_{|S_p}) \geq \Phi_i(v_{|S_k})$ for every $p \in \{k+1, \ldots, |N|\}$ and $\Phi_{i_m}(v_{|S_p}) \geq 0$, for all $m \in \{k+1, \ldots, |N|\}$ and for all $p \in \{m, \ldots, |N|\}$. Hence $\Phi_{i_m}(v_{|S_p}) = 0$ for all $m \in \{k+1, \ldots, |N|\}$ and for all $p \in \{m, \ldots, |N|\}$ and $\Phi_i(v_{|S_k}) = \Phi_i(v_{|S_p})$ for all $p \in \{k+1, \ldots, |N|\}$ and for all $i \in S_k$.

(R3) Suppose on the contrary that $MWC(v_{|S_k}) \neq MWC(v)$. Then there exists a $T \in MWC(v)$ such that $T\backslash S_k \neq \emptyset$. But then $\Phi_j(v) > 0$ for every $j \in T\backslash S_k$, a contradiction with (R1). □

We now provide a characterization of the family of simple games which allow for population monotonic Shapley path schemes.

**Theorem 3.1** *Let* $v \in \mathcal{S}^N$*. Then* $v$ *has a population monotonic Shapley path scheme if and only if* $v$ *is balanced.*

*Proof* Let $v \in \mathcal{S}^N$ be a simple game which has population monotonic Shapley path schemes. Let $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ be a population monotonic Shapley path scheme for $v$ such that $\mathbb{S} = \{S_1, S_2, \ldots, S_{|N|}\}$ and $S_m = \{i_1, \ldots, i_m\}$ for every $m \in \{1, \ldots, |N|\}$. Assume that the first winning coalition along the path $\mathbb{S}$ is $S_k$. Obviously $i_k \in veto(v_{|S_k})$ and hence $veto(v_{|S_k}) \neq \emptyset$. Moreover, we know by (R3) in Lemma 3.1 that $MWC(v_{|S_k}) = MWC(v)$. Hence, $veto(v_{|S_k}) = veto(v)$ and $v$ is balanced.

Now, assume that $v$ is balanced. Then, $veto(v) \neq \emptyset$. Let $i \in veto(v)$ and consider a path $\mathbb{S} = \{S_1, S_2, \ldots, S_{|N|}\}$ with $S_{|N|-1} = N\backslash\{i\}$. We know that $S_{|N|-1} = N\backslash\{i\}$ is a losing coalition. Then $v_{|N\backslash\{i\}}$ is a null game and hence $\Phi_j(v_{|S_t}) = 0$ for all $t \in \{1, \ldots, |N|-1\}$ and $j \in S_t$. Also, $\Phi_j(v) \geq 0$ for all $j \in N$, since $v$ is monotonic. So, the Shapley path scheme $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ is population monotonic. □

Theorem 3.1 reveals that, in the class of simple games, the existence of veto players (or, equivalently, a nonempty core) is a must for the existence of population monotonic Shapley path schemes and vice versa. We can interpret this result as follows. When a winning coalition is formed through a sequence of binding bilateral agreements, we know that the restriction of the TU-game to this coalition has veto players, that is, in this winning coalition, there is a subgroup of agents whose unanimous agreement/involvement is necessary to pass a decision. We also know that the formation of the grand coalition starting from this winning coalition via binding bilateral agreements requires the remaining players to be null players. But, this in turn implies that the veto players of the winning coalition are in fact the veto players of the whole game, i.e., the game is balanced.

Next we turn to our second question of which Shapley path schemes are population monotonic. We will show in the following theorem that the requirement that the first winning coalition along a path has to include all minimum winning coalitions of the game is both necessary and sufficient for the population monotonicity of the corresponding Shapley path scheme.

**Theorem 3.2** *Let $v \in \mathcal{S}^N$ be balanced. A Shapley path scheme $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ is population monotonic if and only if the first winning coalition along $\mathbb{S}$ contains every minimal winning coalition of $v$.*

*Proof* Let $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ be a population monotonic Shapley path scheme for $v$ and assume that the first winning coalition along the path $\mathbb{S}$ is $S_k$. We already know by (R3) in Lemma 3.1 that $MWC(v_{|S_k}) = MWC(v)$. Then, clearly, $S_k$ contains every minimal winning coalition of $v$.

Let $\mathbb{S}$ be a path of coalitions with $S_m = \{i_1, \ldots, i_m\}$ for every $m \in \{1, \ldots, |N|\}$. Assume that the first winning coalition along the path $\mathbb{S}$ is $S_k$ $(k \in \{1, \ldots, |N|\})$ and $S_k$ contains every minimal winning coalition of $v$. Now, $\Phi_j(v_{|S_t}) = 0$ for all $t \in \{1, \ldots, k-1\}$ and $j \in S_t$ since $S_{k-1}$ is a losing coalition. Also, $\Phi_i(v_{|S_k}) \geq 0$ for all $i \in S_k$ since $v$ is monotonic. We know that each player $i_m$ $(m \in \{k+1, \ldots, |N|\})$ is a null player in $v_{|S_p}$ $(p \in \{m, \ldots, |N|\})$ since $S_k$ contains every minimal winning coalition of $v$. Then, firstly, $\Phi_{i_m}(v_{|S_p}) = 0$ for all $m \in \{k+1, \ldots, |N|\}$ and for all $p \in \{m, \ldots, |N|\}$ and secondly, one can easily show that $\Phi_i(v_{|S_k}) = \Phi_i(v_{|S_{k+1}}) = \cdots = \Phi_i(v)$ for all $i \in S_k$ by applying the null player out property recursively. So, we conclude that the Shapley path scheme $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ is population monotonic. □

In the light of Theorem 3.2, we can answer one other important question in this context: For which simple games all Shapley path schemes are population monotonic, i.e., which simple games have a population monotonic Shapley allocation scheme?

**Theorem 3.3** *Let $v \in \mathcal{S}^N$ be a simple game. Then the following statements are equivalent:*

*(i) All Shapley path schemes of $v$ are population monotonic.*
*(ii) The set of veto players of $v$ is a winning coalition.*
*(iii) The game $v$ is convex.*
*(iv) The Shapley allocation scheme of $v$ is population monotonic.*

*Proof* $(i) \to (ii)$ Assume that all Shapley path schemes of $v$ are population monotonic. Suppose that $veto(v)$ is losing. Then there exists a minimum winning coalition $S = \{i_1, \ldots, i_m\}$ with $m \in \{1, ..., |N| - 1\}$. We know that $\Phi_i(v_{|S}) = \frac{1}{m}$ for every $i \in S$ since $S$ is a minimal winning coalition. Pick a path of coalitions $\mathbb{S} = \{S_1, S_2, \ldots, S_{|N|}\}$

with $S_m = S$. The Shapley path scheme $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ is population monotonic by assumption. Consequently, $\Phi_i(v) = \frac{1}{m}$ for every $i \in S$. Observe that there exists $i^* \in S$ such that $i^* \notin veto(v)$ since $S$ is a minimal winning coalition and $veto(v)$ is losing. Then, there exists another minimal winning coalition $T \subsetneq N$ such that $i^* \notin T$. Pick a path of coalitions $\mathbb{S}^{'} = \{S_1^{'}, S_2^{'}, \ldots, S_{|N|}^{'}\}$ with $S_{|T|}^{'} = T$. Now, the Shapley path scheme $(\mathbb{S}', (\Phi(v_{|S}))_{S\in\mathbb{S}'})$ is also population monotonic by assumption. Then, $(R1)$ implies that $\Phi_{i^*}(v) = 0$ since $i^* \notin T$, a contradiction with $\Phi_{i^*}(v) = \frac{1}{m}$ as derived earlier.

$(ii) \rightarrow (iii)$ Let $v \in \mathcal{S}^N$ be such that $veto(v)$ is a winning coalition. Then, all players in $N \backslash veto(v)$ are null players in $v$. Hence, $v$ is the unanimity game on $veto(v)$ and is convex.
$(iii) \rightarrow (iv)$ See Sprumont (1990), Corollary 2.
$(iv) \rightarrow (i)$ Obvious. □

Theorem 3.3 reveals that, in the class of simple games, the existence of a winning veto player set is both necessary and sufficient for the existence of a population monotonic Shapley allocation scheme. This result can be interpreted by making use of our results on population monotonic Shapley path schemes as follows. We know that the existence of a population monotonic Shapley allocation scheme implies the population monotonicity of each Shapley path scheme of the game and vice versa. Then, by Theorem 3.2, the existence of a population monotonic Shapley allocation scheme requires the first winning coalition along each path to include all minimum winning coalitions of the game. But this is possible only when the game has a unique minimum winning coalition, i.e., when the set of veto players is winning.

## 4 Extensions to efficient probabilistic values

Probabilistic values, introduced and characterized by Weber (1988), are generalizations of the Shapley value for finite TU-games. These values keep one essential feature of the Shapley value, they assign each player an average of his marginal contributions. They, however, may fail to satisfy the efficiency property. We refer to Monderer and Samet (2002) for a detailed discussion of probabilistic values.

Probabilistic values are formally defined as follows. Given $N$ and $i \in N$, let $P_N^i$ denote the set of probability distributions on $2^{N\backslash\{i\}}$, the family of coalitions not containing $i$. A value $F$ (defined on $\mathcal{G}^N$) is called a *probabilistic value* (Weber 1988) if for every $v \in \mathcal{G}^N$ and $i \in N$

$$F_i(v) = \sum_{T\subset N\backslash\{i\}} p^i(T)\,(v(T\cup\{i\}) - v(T))\,, \tag{2}$$

for some $p^i \in P_N^i$ for all $i \in N$. Here $p^i \in P_N^i$ can be interpreted as the player's subjective evaluation of the probability of joining different coalitions. For example, the probabilistic value which is defined by $p^i(T) = \frac{1}{|N|}\binom{|N|-1}{|T|}^{-1}$ for all $i \in N$ is the Shapley value.

In the remaining of this article, we will discuss the extensions of the results obtained for the Shapley value on efficient probabilistic values.

Let $\mathcal{P}(\Pi(N))$ denote the set of probability distributions on the set of permutations of the player set $N$. Given $i \in N$ and $S \in 2^{N\setminus\{i\}}$, we will denote by $\Pi^{S,i}(N)$ the set

$$\{\tau \in \Pi(N)|\tau(j) < \tau(i) \text{ if and only if } j \in S\}.$$

If we think of a permutation $\tau \in \Pi(N)$ as the order in which players enter the game, then $\Pi^{S,i}(N)$ stands for the set of orders in which exactly all members of $S$ enter the game before player $i$ enters.

The following characterization of efficient probabilistic values is provided by Weber (1988).

**Theorem 4.1** (Weber 1988) *Let $F$ be a probabilistic value as given in (2) defined by $p=\{p^i\}_{i\in N}$ with $p^i \in P^i_N$ for every $i \in N$. Then $F$ is efficient if and only if there exists $b \in \mathcal{P}(\Pi(N))$ such that*

$$p^i(S) = \sum_{\tau\in\Pi^{S,i}(N)} b(\tau) \tag{3}$$

*for every $i \in N$ and $S \in 2^{N\setminus\{i\}}$.*

Observe that probabilistic values are originally defined for a fixed player set. However, our analysis requires the values to be defined on every subset of the player set under consideration because, for every simple game, we want to be able to compare the payoffs assigned by a value to the players at every subgame of the game. We now extend probabilistic values in such a way that the players' subjective evaluation of the probability of joining different coalitions will be consistent in the sense defined below. For this aim we will define the *restrictions* of a probabilistic value to subgames.

Let $F : \mathcal{G}^N \to \mathbb{R}^N$ be a probabilistic value defined by $\{p^i_N\}_{i\in N}$ where $p^i_N \in P^i_N$ for every $i \in N$. For each $S \subset N$, the *restriction* of $F$ to $\mathcal{G}^S$ is denoted by $F_S$ and for each player $i \in S$, his restricted evaluations $p^i_S \in P^i_S$ are constructed by using the following consistency condition.

$$p^i_S(T) = \sum_{T'\subset N\setminus S} p^i_N(T \cup T'), \tag{4}$$

for all $T \subset S\setminus\{i\}$.

We first illustrate the notion of the restriction of a probabilistic value in the following example.

**Example 4.1** Let $F$ be a probabilistic value on $N = \{1, 2, 3\}$. Assume that $F$ is defined by the following subjective evaluations of players.

$$p_N^1(\{2,3\}) = \frac{5}{16}, p_N^1(\{2\}) = \frac{1}{16}, p_N^1(\{3\}) = \frac{4}{16} \text{ and } p_N^1(\emptyset) = \frac{6}{16}.$$
$$p_N^2(\{1,3\}) = \frac{8}{16}, p_N^2(\{1\}) = \frac{2}{16}, p_N^2(\{3\}) = \frac{4}{16} \text{ and } p_N^2(\emptyset) = \frac{2}{16}.$$
$$p_N^3(\{1,2\}) = \frac{3}{16}, p_N^3(\{1\}) = \frac{4}{16}, p_N^3(\{2\}) = \frac{1}{16} \text{ and } p_N^3(\emptyset) = \frac{8}{16}.$$

$F$ satisfies (3) by taking the following probability distribution on the set of permutations on the player set:

$$b(123) = \frac{2}{16}, b(132) = \frac{4}{16}, b(213) = \frac{1}{16}, b(231) = \frac{1}{16}, b(312) = \frac{4}{16}, \text{ and}$$
$$b(321) = \frac{4}{16}.$$

Hence $F$ is efficient.

Now consider $S = \{1, 2\}$. According to (4), the restriction $F_S$ is defined by

$$p_S^1(\{2\}) = \frac{3}{8} = p_N^1(\{2\}) + p_N^1(\{2,3\}) \quad \textit{and} \quad p_S^1(\emptyset) = \frac{5}{8} = p_N^1(\emptyset) + p_N^1(\{3\}).$$
$$p_S^2(\{1\}) = \frac{5}{8} = p_N^2(\{1\}) + p_N^2(\{1,3\}) \quad \textit{and} \quad p_S^2(\emptyset) = \frac{3}{8} = p_N^2(\emptyset) + p_N^2(\{3\}).$$

Notice that $F_S$ can be described via (3) by taking

$$b(12) = \frac{5}{8} \text{ and } b(21) = \frac{3}{8}.$$

So $F_S$ is an efficient probabilistic value on $\mathcal{G}^S$.

In the previous example, we have shown that the specific restriction under consideration is again an efficient probabilistic value. Indeed, every restriction of an efficient probabilistic value is an efficient probabilistic value for the corresponding subgame as shown in the following proposition.

**Proposition 4.1** *Let $F$ be an efficient probabilistic value defined by $\{p_N^i\}_{i\in N}$ where $p_N^i \in P_N^i$ for every $i \in N$. Then, $F_S$ is an efficient probabilistic value for every $S \subset N$, $S \neq \emptyset$.*

*Proof* By Theorem 4.1 there exists $b \in \mathcal{P}(\Pi(N))$ such that $p_N^i(T) = \sum_{\tau\in\Pi^{T,i}(N)} b(\tau)$ for every $i \in N$ and $T \in 2^{N\setminus\{i\}}$. Take $S \subset N$, $S \neq \emptyset$. Given $\tau \in \Pi(N)$, $\tau_{|S}$ denotes the restriction of $\tau$ to $S$, i.e., $\tau_{|S} = \pi$ for some $\pi \in \Pi(S)$ with $\pi(i) < \pi(j)$ if and only if $\tau(i) < \tau(j)$, for all $i, j \in S$. We can induce a probability distribution $c$ on $\Pi(S)$ from $b$ as follows.

$$c(\pi) = \sum_{\tau\in\Pi(N):\tau_{|S}=\pi} b(\tau), \quad \text{for all } \pi \in \Pi(S). \tag{5}$$

Let $F_S$ be defined by $\{p_S^i\}_{i\in S}$ as determined by (4). Pick $i \in S$ and $T \subset S\backslash\{i\}$. Obviously,

$$\bigcup_{T'\subset N\backslash S} \Pi^{(T\cup T'),i}(N) = \bigcup_{\pi\in\Pi^{T,i}(S)} \{\tau \in \Pi(N)|\tau_{|S} = \pi\}. \tag{6}$$

Notice that

$$\Pi^{(T\cup T'),i}(N) \cap \Pi^{(T\cup T''),i}(N) = \emptyset \quad \text{for every } T', T'' \subset N\backslash S \text{ with } T' \neq T''$$

and

$$\{\tau \in \Pi(N)|\tau_{|S} = \pi\} \cap \{\tau \in \Pi(N)|\tau_{|S} = \pi'\} = \emptyset$$
$$\text{for every } \pi, \pi' \in \Pi^{T,i}(S) \text{ with } \pi \neq \pi'.$$

Then,

$$\begin{aligned} p_S^i(T) &= \sum_{T'\subset N\backslash S} p_N^i(T\cup T') \\ &= \sum_{T'\subset N\backslash S} \sum_{\tau\in\Pi^{(T\cup T'),i}(N)} b(\tau) \\ &= \sum_{\pi\in\Pi^{T,i}(S)} \sum_{\tau\in\Pi(N):\tau_{|S}=\pi} b(\tau) \\ &= \sum_{\pi\in\Pi^{T,i}(S)} c(\pi) \end{aligned}$$

where the first equality follows from (4) and the last but one equality follows from (6) and the remarks below it. Then, Theorem 4.1 implies that $F_S$ is an efficient probabilistic value on $\mathcal{G}^S$. □

Having defined the restrictions of probabilistic values, we can now illustrate the path schemes associated with these values in the following example.

**Example 4.2** Consider the probabilistic value $F$ defined in Example 4.1 and let $v \in \mathcal{S}^N$ with $N = \{1, 2, 3\}$ be defined by $MWC(v) = \{\{1, 2\}, \{2, 3\}\}$. From Table 2 it can easily be observed that this balanced game has two population monotonic $F$-path schemes related to the paths {{1},{1,3},N} and {{3},{1,3},N}.

The following theorem states that the results for population monotonic Shapley path schemes in fact can be extended to all efficient probabilistic values which are defined by strictly positive subjective evaluations of joining different coalitions for each player.

**Theorem 4.2** *Let $F : \mathcal{G}^N \to \mathbb{R}^N$ be an efficient probabilistic value defined by $\{p_N^i\}_{i\in N}$ with $p_N^i > 0$ for all $i \in N$. Then*

**Table 2** The restrictions of $F$ for $v$ and its subgames in Example 4.2

| Coalition | Player 1 | Player 2 | Player 3 |
|---|---|---|---|
| {1} | 0 | – | – |
| {2} | – | 0 | – |
| {3} | – | – | 0 |
| {1,2} | 3/8 | 5/8 | – |
| {1,3} | 0 | – | 0 |
| {2,3} | – | 6/8 | 2/8 |
| $N$ | 1/16 | 14/16 | 1/16 |

**Table 3** The restrictions of $F$ for $v$ and its subgames in Example 4.3

| Coalition | Player 1 | Player 2 | Player 3 |
|---|---|---|---|
| {1} | 0 | – | – |
| {2} | – | 0 | – |
| {3} | – | – | 0 |
| {1,2} | 1/2 | 1/2 | – |
| {1,3} | 1/2 | – | 1/2 |
| {2,3} | – | 1/2 | 1/2 |
| $N$ | 1/2 | 1/2 | 0 |

1. *A simple game $v \in \mathcal{S}^N$ has a population monotonic $F$-path scheme if and only if $v$ is balanced.*
2. *Let $v$ be balanced. Then an $F$-path scheme $(\mathbb{S}, (F_S(v_{|S}))_{S\in\mathbb{S}})$ is population monotonic if and only if the first winning coalition along $\mathbb{S}$ contains every minimal winning coalition of $v$.*
3. *Let $v \in \mathcal{S}^N$ be a simple game. Then the following statements are equivalent:*
   *(a) All $F$-path schemes of $v$ are population monotonic.*
   *(b) The set of veto players of $v$ is a winning coalition.*
   *(c) The game $v$ is convex.*
   *(d) The $F$-allocation scheme of $v$, $(F_S(v_{|S}))_{S\in 2^N\setminus\{\emptyset\}})$ is population monotonic.*

The proof of Theorem 4.2 is similar to the proofs of Theorems 3.1, 3.2 and 3.3, respectively and is therefore omitted.

It is important at this point to observe that if for an efficient probabilistic value $F$, $p_N^i(S)=0$ for some $i \in N$ and $S \in 2^{N\setminus\{i\}}$, then an unbalanced simple game may have population monotonic $F$-path schemes. This is illustrated in Example 4.3.

**Example 4.3** Let $N=\{1,2,3\}$. Let $F$ be the efficient probabilistic value determined by

$$p_N^1(S)=\frac{1}{4}\quad \text{for all } S\subset N\backslash\{1\};\ p_N^2(S)=\frac{1}{4}\quad \text{for all } S\subset N\backslash\{2\}\quad \text{and}$$
$$p_N^3(\{1,2\})=p_N^3(\emptyset)=\frac{1}{2},\ p_N^3(\{1\})=p_N^3(\{2\})=0.$$

Consider $v \in \mathcal{S}^N$ defined by $MWC(v) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Clearly $veto(v) = \emptyset$. However, $v$ has population monotonic $F$-path schemes related to the paths $\{\{1\}, \{1, 2\}, N\}$ and $\{\{2\}, \{1, 2\}, N\}$ as can be seen in Table 3.

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